Unit-4

Statistical Techniques - II

Random Variable: A random variable is any numerical quantity whose value will be determined by the outcome of a random experiment and which have a specific range and a definable probability associated with each value.

Or, if the numerical values assumed by a variable are the result of some chance factors, such that a particular value cannot be exactly predicted in advance, the variable is then called a random variable.

A random variable is also called 'chance variable' or 'stochastic variable'. They are usually denoted by capital letters like - X, Y, Z, ...

eg - Throwing of a dice.

Output will be any number from 1 to 6. but it is not fixed. Probability of each is $\frac{1}{6}$ (fixed) so, the output (X) is a random variable.

eg - Tossing of 3 coins simultaneously.

Suppose we have to find the number of heads. different posibilities are

{TTT, HTT, THT, TTH, THH, HTH, HHT, HHH}

So, total posibilities are 8.

probability for 0 heads — 1/8
probability for 1 head — 3/8
probability for 2 heads — 3/8
probability for 3 heads — 1/8

Output can be any of the above, it is not fixed. Probability of each is fixed.

So, the output (X) is a random variable.

Types of Random variables

There are two types of random variables -

(1) **Discrete Random variable** - A discrete random variable is one which can assume only isolated values.

eg - The number of heads in tossing 3 coins simultaneously. The discrete random variable (X) can assume values {0, 1, 2, 3} and no other.

(2) **Continous Random variable** - A continous random variable is one which can assume any value within an interval i.e. all values of a continous scale.

eg - The weights (in kg) of a group of individuals. Suppose the weight lies anywhere between 40 - 100 kg But the value will not be fixed. It can be anything 52.5 or even 80.

Probability Mass Function

Let $x_1, x_2, x_3, \ldots$ be the values of a discrete random variable X and let $p_1, p_2, p_3, \ldots$ be the corresponding probabilities.

$p_i > 0, \quad \forall \; i = 1, 2, 3, \ldots$

Then the probability function $P(X = x)$ or $p(x)$ is given by

$$P(X = x) = p(x) = \begin{cases} p(x_i) \text{ or } p_i, & \text{where } i = 1, 2, 3, \ldots \\ 0, & \text{otherwise.} \end{cases}$$

So, the probability function $p(x)$ gives the probability that the random variable X assumes for any particular value x in its range.

A table or a formula listing all possible values that a random variable can take on together with the respective probabilities, is called a probability distribution of the random variable. i.e. set of ordered pair $[x_i, p(x_i)]$

eg - Tossing of 3 coins simultaneously.
Suppose we have to find the number of heads, for which different possibilities are -
{TTT, HTT, THT, TTH, THH, HTH, HHT, HHH}

X	0	1	2	3
$P(X=x) = p(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

(The above table is the probability distribution of the random variable X)

Mean and Variance

Let

X	x_1	x_2	x_3	...	x_n
$P(X=x)$	p_1	p_2	p_3	...	p_n

be the discrete probability distribution then.

Mean is given by (expected value)

$$\mu = E(X) = \Sigma p_i x_i$$

and variance is given by, $\sigma^2 = \Sigma p_i x_i^2 - \mu^2$

Standard deviation, $\sigma = +\sqrt{\text{Variance}}$

2019

Ques If the p.m.f. of a discrete random variable X is

X	1	2	3
$f(x)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

Determine $E(X)$ and $V(x)$

Sol^n

$X(x_i)$	$f(x)$	$x_i f_i(x)$	$f_i \cdot x_i^2$
1	$\frac{1}{2}$	$1/2$	$1/2$
2	$\frac{1}{3}$	$2/3$	$4/3$
3	$\frac{1}{6}$	$1/2$	$3/2$
		$\frac{5}{3}$	$\frac{10}{3}$

Mean, $\mu = E(X) = \Sigma x_i f_i(x) = \frac{5}{3}$

Variance, $V(x) = \Sigma p_i x_i^2 - \mu^2$

$$= \frac{10}{3} - \frac{25}{9} = \frac{5}{9}$$

Ques Five defective bulbs are accidentally mixed with twenty good ones. It is not possible to just look at a bulb and tell whether or not it is defective. Find the probability distribution of the number of defective bulbs, if four bulbs are drawn at random from this lot.

Sol: Let X denote the number of defective bulbs in 4. X can take the values 0, 1, 2, 3, 4

No. of defective bulbs = 5
No. of good bulbs = 20
Total No. of bulbs = 25.

$P(\text{no defective}) = P(X=0) = P(\text{all 4 good bulbs})$

$$= \frac{^{20}C_4}{^{25}C_4} = \frac{20 \times 19 \times 18 \times 17}{25 \times 24 \times 23 \times 22} = \frac{969}{2530}$$

$P(X=1) = P(\text{one defective \& 3 good bulbs})$

$$= \frac{^5C_1 \times {^{20}C_3}}{^{25}C_4} = \frac{1140}{2530}$$

$P(X=2) = P(\text{two defective \& 2 good bulbs})$

$$= \frac{^5C_2 \times {^{20}C_2}}{^{25}C_4} = \frac{380}{2530}$$

$P(X=3) = P(\text{3 defective \& 1 good bulb})$

$$= \frac{^5C_3 \times {^{20}C_1}}{^{25}C_4} = \frac{40}{2530}$$

$P(X=4) = P(\text{all 4 defective bulbs})$

$$= \frac{^5C_4}{^{25}C_4} = \frac{1}{2530}$$

The probability of the random variable X is

X	0	1	2	3	4
P(X)	$\frac{969}{2530}$	$\frac{1140}{2530}$	$\frac{380}{2530}$	$\frac{40}{2530}$	$\frac{1}{2530}$

Ques A die is tossed thrice. A success is 'getting 1 or 6' on a toss. Find the mean and the variance of the number of successes.

Sol^n Let X denote the number of successes.

X can take the value 0, 1, 2, 3

Probability of success (getting 1 or 6) $= \frac{2}{6} = \frac{1}{3}$

Probability of failure $= 1 - \frac{1}{3} = \frac{2}{3}$

$P(X=0) = P$(no success or all 3 failures)

$= {}^3C_0 \times \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{8}{3}$

$P(X=1) = P$(1 success & 2 failures)

$= {}^3C_1 \times \frac{1}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{12}{27}$

$P(X=2) = P$(2 successes & 1 failure)

$= {}^3C_2 \times \frac{1}{3} \times \frac{1}{3} \times \frac{2}{3} = \frac{6}{27}$

$P(X=3) = P$(all 3 successes)

$= {}^3C_3 \times \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} = \frac{1}{27}$

The probability distribution of the random variable X is

X	0	1	2	3
P(X)	$\frac{8}{27}$	$\frac{12}{27}$	$\frac{6}{27}$	$\frac{1}{27}$

For mean & variance,

x_i	p_i	$p_i x_i$	$p_i x_i^2$
0	8/27	0	0
1	12/27	12/27	12/27
2	6/27	12/27	24/27
3	1/27	3/27	9/27
		1	5/3

Mean, $\mu = \Sigma p_i x_i = 1$

Variance, $\sigma^2 = \Sigma p_i x_i^2 - \mu^2 = \frac{5}{3} - 1 = \frac{2}{3}$

Ques Find the probability distribution of the number of doublets in four throws of a pair of dice.

Sol.ⁿ If we throw pair of dice, the possibilities are

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

Total possible outcomes are = 36

doublets are (1,1) (2,2) (3,3) (4,4) (5,5) (6,6)

no. of doublets = 6

let X denote number of successes i.e. no. of doublets

X can take the value 0, 1, 2, 3, 4

Probability of success $= \frac{6}{36} = \frac{1}{6}$

Probability of failure $= 1 - \frac{1}{6} = \frac{5}{6}$

$P(X=0) = P$(no doublet or all 4 failure)

$$= {}^4C_0 \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} = \frac{625}{1296}$$

$P(X=1) = P(1 \text{ success } \& \text{ 3 failures})$

$$= {}^4C_1 \times \frac{1}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} = \frac{500}{1296}$$

$P(X=2) = P(2 \text{ success } \& \text{ 2 failure})$

$$= {}^4C_2 \times \frac{1}{6} \times \frac{1}{6} \times \frac{5}{6} \times \frac{5}{6} = \frac{150}{1296}$$

$P(X=3) = P(3 \text{ success } \& \text{ 1 failure})$

$$= {}^4C_3 \times \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} \times \frac{5}{6} = \frac{20}{1296}$$

$P(X=4) = P(4 \text{ success})$

$$= {}^4C_4 \times \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} = \frac{1}{1296}$$

The probability distribution for random variable X is

X	0	1	2	3	4
P(X)	$\frac{625}{1296}$	$\frac{500}{1296}$	$\frac{150}{1296}$	$\frac{20}{1296}$	$\frac{1}{1296}$

Ques Two cards are drawn simultaneously from a well-shuffled deck of 52 cards. Compute the variance for the number of aces.

[Ans: $\sigma^2 = \frac{400}{2873}$]

Probability Density Function

For a density function $f(x)$, the probability that the variate X falls in any interval (p, q) is given by,

$$P(p \le X < q) = \int_p^q f(x)\,dx$$

The density function possesses the following two properties:

(1) $f(x) \geq 0$, for every x, as negative probability has no meaning.

(2). $\int_{-\infty}^{\infty} f(x)\,dx = 1$, this corresponds to the fact that the probability of an event that is sure to happen is equal to unity. (Total probability is 1)

Mean and Variance:

$$\text{Mean} = \int_a^b x\, f(x)\,dx$$

$$\text{Variance} = \int_a^b x^2 f(x)\,dx - (\text{Mean})^2$$

2019
Ques The probability density function $f(x)$ of a continous random variable X is defined by

$$f(x) = \begin{cases} \frac{A}{x^2}, & 5 \le x \le 10 \\ 0, & \text{otherwise} \end{cases}$$

. Find the value of A

Sol.) We know that the total probability is 1

i.e. $\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \int_5^{10} \frac{A}{x^2}\,dx = 1$

$$A \int_5^{10} \frac{1}{x^2}\,dx = 1$$

$$\Rightarrow A\left[-\frac{1}{x}\right]_5^{10} = 1$$

$$\Rightarrow -A\left[\frac{1}{10}-\frac{1}{5}\right] = 1 \Rightarrow -A\left[\frac{1-2}{10}\right] = 1$$

$$\Rightarrow A = 10$$

Ques Let X be a continous random variable with p.d.f. given by

$$f(x) = \begin{cases} kx, & 0 \le x < 1 \\ k, & 1 \le x < 2 \\ -kx+3k, & 2 \le x < 3 \\ 0, & \text{elsewhere.} \end{cases}$$

Determine the constant k.

Sol. We know $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_0^3 f(x)\,dx = 1$$

$$\int_0^1 kx\,dx + \int_1^2 k\,dx + \int_2^3 (-kx+3k)\,dx = 1$$

$$k\left[\frac{x^2}{2}\right]_0^1 + k\left[x\right]_1^2 + k\left[-\frac{x^2}{2}+3x\right]_2^3 = 1$$

$$\Rightarrow k\left[\frac{1}{2}\right] + k\left[2-1\right] + k\left[-\frac{9}{2}+9+2-6\right] = 1$$

$$\Rightarrow k\left[\frac{1}{2}\right] + k\left[1\right] + k\left[\frac{1}{2}\right] = 1$$

$$\Rightarrow k = \frac{1}{2}$$

Ques A random variable X is distributed at random between the value 0 and 1 so that its probability density function is $f(x) = kx^2(1-x^3)$, where k is a constant. Find the value of k. Using this value of k, find its mean and variance.

Sol^n We have $\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \int_0^1 f(x)\,dx = 1$

$$\Rightarrow k\int_0^1 (x^2 - x^5)\,dx = 1$$

$$\Rightarrow k\left[\frac{x^3}{3} - \frac{x^6}{6}\right]_0^1 = 1 \Rightarrow k = 6$$

$$\text{Mean} = \int_a^b x f(x)\,dx$$

$$= 6\int_0^1 x(x^2 - x^5)\,dx = 6\int_0^1 (x^3 - x^6)\,dx$$

$$= 6\left[\frac{x^4}{4} - \frac{x^7}{7}\right]_0^1 = \frac{9}{14}$$

$$\text{Variance} = \int_a^b x^2 f(x)\,dx - (\text{Mean})^2$$

$$= 6\int_0^1 x^2(x^2 - x^5)\,dx - \left(\frac{9}{14}\right)^2$$

$$= 6\left[\frac{x^5}{5} - \frac{x^8}{8}\right]_0^1 - \left(\frac{9}{14}\right)^2 = \frac{9}{20} - \left(\frac{9}{14}\right)^2 = \frac{9}{245}$$

Ques A variable X is distributed at random between the values 0 and 4 and its probability density function is given by: $f(x) = kx^3(4-x)^2$. Find the value of k, the mean and standard deviation of the distribution

[Ans: $k = \frac{15}{1024}$, Mean $= \frac{16}{7}$, Variance $= \frac{24}{49}$

Standard deviation $= \sqrt{\text{Variance}} = \frac{2\sqrt{6}}{7}$]

Ques A continous random variable X has a p.d.f $f(x) = 3x^2$, $0 \le x \le 1$. Find 'a' and 'b' such that (i) $P(X \le a) = P(X > a)$ and (ii) $P(X > b) = 0.05$

Sol.n (i) Since $P(X \le a) = P(X > a)$, so each must be equal to $\frac{1}{2}$ as total probability is 1.

$$P(X \le a) = \frac{1}{2}$$

$$\int_0^a f(x)\,dx = \frac{1}{2}$$

$$\Rightarrow 3\int_0^a x^2\,dx = \frac{1}{2}$$

$$\Rightarrow 3\left[\frac{x^3}{3}\right]_0^a = \frac{1}{2} \Rightarrow a = \left(\frac{1}{2}\right)^{1/3}$$

(ii)

$$P(X > b) = 0.05$$

$$\int_b^1 f(x)\,dx = 0.05$$

$$\Rightarrow 3\int_b^1 x^2\,dx = 0.05$$

$$\Rightarrow 3\left[\frac{x^3}{3}\right]_b^1 = \frac{1}{20} \Rightarrow 1 - b^3 = \frac{1}{20}$$

$$\Rightarrow b = \left(\frac{19}{20}\right)^{1/3}$$

Ques If $f(x) = \begin{cases} 2x, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$, find the probability $P\left(\frac{1}{4} \le x \le \frac{1}{2}\right)$

Sol.n Since $\int_{-\infty}^{\infty} f(x)\,dx$ will give the required probability

$$P\left(\frac{1}{4} \le x \le \frac{1}{2}\right) = \int_{1/4}^{1/2} f(x)\, dx$$

$$= 2\int_{1/4}^{1/2} x\, dx = \frac{3}{16}$$

Ques Find the constant k so that func^n f(x) defined as follows be a density func^n

$$f(x) = \begin{cases} 1/k, & a \le x \le b \\ 0, & \text{elsewhere.} \end{cases}$$

[Ans: $k = b - a$]

Ques The kms. X (in thousands of kms) which car owners get with a certain kind of tyre is a random variable having probability density function:

$$f(x) = \begin{cases} \frac{1}{20} e^{-x/20}, & \text{for } x > 0 \\ 0, & \text{for } x \le 0 \end{cases}$$

Find the probabilities that one of these tyres will last (i) atmost 10,000 kms (ii) anywhere from 16,000 to 24,000 kms and (iii) atleast 30,000 miles.

Sol.^{n} (i) atmost 10,000 kms

$$\Rightarrow P(X \le 10) = \int_{0}^{10} f(x)\, dx = \int_{0}^{10} \frac{1}{20} e^{-x/20}\, dx$$

$$= \frac{1}{20}\left[\frac{e^{-x/20}}{-1/20}\right]_0^{10} = \left[1 - e^{-0.5}\right] = 0.3935$$

(ii) anywhere from 16,000 to 24,000 kms

$$P(16 \le X \le 24) = \int_{16}^{24} \frac{1}{20} e^{-x/20}\, dx$$

$$= \frac{1}{20}\left[\frac{e^{-x/20}}{-1/20}\right]_{16}^{24}$$

$$= \left[e^{-4/5} - e^{-6/5}\right] = 0.1481$$

(iii) atleast 30,000 miles

$$P(X \geq 30) = \int_{30}^{\infty} f(x)\,dx$$

$$= \frac{1}{20}\int_{30}^{\infty} e^{-x/20}\,dx$$

$$= \frac{1}{20}\left[\frac{e^{-x/20}}{-1/20}\right]_{30}^{\infty} = -\left[e^{-\infty} - e^{-3/2}\right]$$

$$= e^{-3/2} = 0.2231 \qquad \left[e^{-\infty} = 0\right]$$

Theoretical Probability Distributions

Probability Distributions

- Observed frequency Distribution (based on actual observation & experimentation)
- Theoretical or Expected frequency Distribution (based on hypothesis, which may or ~~not~~ may not be derived mathematically)
  - (Discrete probability distribution)
    - Binomial Distribution
    - Poisson Distribution
  - (Continuous Probability distribution)
    - Normal Distribution.

Binomial Distribution

This distribution is concerned with trials of a repetitive nature in which only the occurrence or non-occurrence (i.e. success or failure) is of interest.

It is a discrete distribution as X can take only the integral values 0, 1, 2, 3, ...

Let there be 'n' independent trials in an experiment. Let a random variable X denote the number of success in these n trials.

Let p be the probability of a success and q that of a failure in a single trial so, that $p+q=1$.

Let the trials be independent and p be constant for every trial

then the probability of 'r' successes in 'n' trials is given by

$$\boxed{P(X=r) = {}^nC_r \, p^r \cdot q^{n-r}}$$

where $p+q=1$ & $r=0,1,2,3 \ldots n$

This is called binomial probability distribution

Conditions for a Binomial distribution

* Each trial result in two mutually disjoint outcomes, success or failure.
* The number of trials n is finite
* The trials are independent to each other.
* The probability of success 'p' is constant for each trial.

Note: n and p occurring in the binomial distribution are called the parameters of the distribution i.e. $B(n,p)$

imp **Moment generating function of Binomial distribution**

$$M_x(t) = \Sigma e^{tx} f(x) \quad \text{[for discrete distribution]}$$

$$= \Sigma e^{tx} \cdot {}^nC_x p^x \cdot q^{n-x}$$

on solving,

$$\boxed{M_x(t) = (q+pe^t)^n}$$

$$\mu_1' = \left| \frac{d}{dt} M_x(t) \right|_{t=0} = np$$

Mean $= \mu_1'$

$$\boxed{\text{Mean} = np}$$

$$\mu_2' = \left| \frac{d^2}{dt^2} M_x(t) \right|_{t=0} = np(np+q)$$

$$\text{Variance} = \mu_2 = \mu_2' - (\mu_1')^2$$

$$\Rightarrow \boxed{\text{Variance} = npq}$$

Recurrence formula for the Binomial distribution

$$P(r) = {}^nC_r p^r q^{n-r} = \frac{n!}{r!(n-r)!} p^r q^{n-r}$$

$$P(r+1) = {}^nC_{r+1} p^{r+1} q^{n-r-1} = \frac{n!}{(r+1)!(n-r-1)!} p^{r+1} q^{n-r-1}$$

dividing,

$$\frac{P(r+1)}{P(r)} = \frac{\cancel{n!}}{(r+1)!(n-r-1)!} \cdot \frac{r!(n-r)!}{\cancel{n!}} \quad \frac{p^{r+1} \cdot q^{n-r-1}}{p^r \, q^{n-r}}$$

$$= \frac{(n-r)!}{(n-r-1)!} \cdot \frac{r!}{(r+1)!} \cdot \frac{p}{q}$$

$$= \frac{(n-r)\cancel{(n-r-1)!}}{\cancel{(n-r-1)!}} \quad \frac{\cancel{r!}}{(r+1)\cancel{r!}} \cdot \frac{p}{q}$$

$$= \frac{n-r}{r+1} \cdot \frac{p}{q}$$

$$\Rightarrow \boxed{P(r+1) = \frac{n-r}{r+1} \cdot \frac{p}{q} \cdot P(r)}$$

Binomial Frequency Distribution

If n independent trials constitute one experiment and this experiment is repeated N-times then the frequency of r successes is $N \cdot {}^nC_r \, p^r \cdot q^{n-r}$

Ques A binomial variable X satisfies the relation $9P(X=4) = P(X=2)$ when $n=6$. Find the value of the parameter p and $P(X=1)$

Sol.n we have $P(X=r) = {}^nC_r \, p^r q^{n-r}$

$\Rightarrow$ $P(X=4) = {}^6C_4 \, p^4 q^2$ (as $n=6$)

$P(X=2) = {}^6C_2 \, p^2 q^4$

the given relation is:

$$9P(X=4) = P(X=2)$$

putting the values in given relation we have,

$9 \times 15 p^4 q^2 = 15 p^2 q^4$ $\quad \left[\begin{array}{l} p+q=1 \\ \Rightarrow q = 1-p \end{array}\right]$

$\Rightarrow 9p^2 = q^2$

$\Rightarrow 9p^2 = (1-p)^2$

$\Rightarrow 9p^2 = 1 + p^2 - 2p$

$\Rightarrow 8p^2 + 2p - 1 = 0$

$\Rightarrow (4p-1)(2p+1) = 0$

$\Rightarrow p = 1/4$, as p cannot be negative

$\Rightarrow q = 1-p = 1 - 1/4 = 3/4$

$\therefore P(X=1) = {}^6C_1 \left(\frac{1}{4}\right)^1 \left(\frac{3}{4}\right)^5 = 0.3559$

Ques If the moment generating function of a random variable X is $\left(\frac{1}{3} + \frac{2}{3}e^t\right)^5$. Find $P(X=2)$

Sol^n we have $M_x(t) = (q + pe^t)^n$

given : $M_x(t) = \left(\frac{1}{3} + \frac{2}{3}e^t\right)^5$

Comparing, $p = \frac{2}{3}$, $q = \frac{1}{3}$, $n = 5$

we have $P(X=x) = {}^nC_x \, p^x \, q^{n-x}$

$\Rightarrow P(X=2) = {}^5C_2 \cdot \left(\frac{2}{3}\right)^2 \cdot \left(\frac{1}{3}\right)^3$

$= \frac{40}{243} = 0.1646$

2018
Ques If the probability of hitting a target is 10% and 10 shots are fired independently. What is the probability that the target will be hit atleast once?

Sol.n probability of success (hitting the target) is

$$p = \frac{10}{100} = \frac{1}{10}$$

$$q = 1 - p = 1 - \frac{1}{10} = \frac{9}{10}$$

Probability that the target will hit atleast once is $= P(r \geq 1)$

$$= P(1) + P(2) + P(3) + \cdots + P(10)$$

$$= 1 - P(0)$$

(as total probability is 1)

$$= 1 - ({}^{n}C_0\, p^0 \cdot q^n)$$

$$= 1 - {}^{10}C_0 \left(\frac{9}{10}\right)^{10} = 1 - \left(\frac{9}{10}\right)^{10}$$

$$= 1 - 0.3486$$

$$\approx 0.6513$$

2016
Ques In 800 families with 5 children each, how many families would be expected to have (i) 3 boys and 2 girls, (ii) 2 boys & 3 girls (iii) no girl (iv) at the most 2 girls (Assume probabilities for boys & girls to be equal)

Sol.n given $n = 5$

Let probability of having a boy, $p = \frac{1}{2}$

probability of having a girl, $q = \frac{1}{2}$

(i) Probability of having 3 boys & 2 girls
i.e. $r=3$

$P(X=r) = {}^nC_r\, p^r\, q^{n-r}$

$P(X=3) = {}^5C_3 \left(\frac{1}{2}\right)^3 \cdot \left(\frac{1}{2}\right)^{5-3} = \frac{10}{32} = 0.3125$

$\therefore$ expected no. of families to have 3 boys & 2 girls is $= 800 \times 0.3125 = 250$

(ii) Probability of having 2 boys & 3 girls
i.e. $r=2$

$P(X=2) = {}^5C_2 \cdot \left(\frac{1}{2}\right)^2 \cdot \left(\frac{1}{2}\right)^{5-2} = \frac{10}{32} = 0.3125$

$\therefore$ expected no. of families to have 2 boys & 3 girls is $= 800 \times 0.3125 = 250$

(iii) Probability of having no girl i.e. all 5 are boys $\Rightarrow r=5$

$P(X=5) = {}^5C_5 \left(\frac{1}{2}\right)^5 \cdot \left(\frac{1}{2}\right)^{5-5} = \left(\frac{1}{2}\right)^5$

$= \frac{1}{32} = 0.03125$

$\therefore$ expected no. of families to have 0 girls is $= 800 \times 0.03125 = 25$

(iv) Probability of having at the most two girls
i.e. 0 girl, 1 girl or 2 girls
or, 5 boys, 4 boys or 3 boys

$P(\text{almost 2 girls}) = P(X=5) + P(X=4) + P(X=3)$

$$= \left[{}^5C_5 \left(\tfrac{1}{2}\right)^5 \cdot \left(\tfrac{1}{2}\right)^0\right] + \left[{}^5C_4 \left(\tfrac{1}{2}\right)^4 \left(\tfrac{1}{2}\right)^1\right] + \left[{}^5C_3 \left(\tfrac{1}{2}\right)^3 \cdot \left(\tfrac{1}{2}\right)^2\right]$$

$$= \frac{1}{32} + \frac{5}{32} + \frac{10}{32} = \frac{16}{32} = \frac{1}{2} = 0.5$$

$\therefore$ expected no. of families to have at the most two girls $= 800 \times 0.5 = 400$

Ques Out of 800 families with 4 children each, how many families would be expected to have (i) 2 boys and 2 girls (ii) at least one boy (iii) no girl (iv) atmost two girls? Assume equal probabilities for boys and girls.

[Ans: (i) 300 (ii) 750 (iii) 50 (iv) 550]

2019

Ques Find the mean of the Binomial Distribution $B(4, \frac{1}{3})$

Sol^n We know n & p are parameters of binomial distribution i.e. $B(n, p)$

So, on comparing $n = 4$, $p = \frac{1}{3}$

$$\text{Mean} = np = 4 \times \frac{1}{3} = \frac{4}{3}$$

Poisson Distribution

Poisson Distribution as a limiting case of Binomial distribution.

If n & p are known we can find the Binomial distribution but if n is very large and p is very small, application of Binomial distribution is very difficult. So if we assume that as $n \to \infty$ and $p \to 0$ such that np always remains finite, say λ, we get the Poisson approximation to the Binomial distribution.

For a binomial distribution,

$$P(X=r) = {}^nC_r p^r q^{n-r} \qquad [\text{as } p+q=1]$$

$$= \frac{n!}{r!.(n-r)!} p^r . (1-p)^{n-r}$$

$$= \frac{n(n-1)(n-2) \cdots (n-r+1)}{r!} p^r (1-p)^{n-r}$$

as $np = \lambda \Rightarrow p = \lambda/n$

$$\Rightarrow P(X=r) = \frac{n(n-1)(n-2) \cdots (n-r+1)}{r!} \left(1-\frac{\lambda}{n}\right)^{n-r} \left(\frac{\lambda}{n}\right)^r$$

$$= \frac{\lambda^r}{r!} . \frac{n(n-1)(n-2) \cdots (n-r+1)}{n^r} \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^r}$$

$$= \frac{\lambda^r}{r!} \frac{n}{n} . \left(\frac{n-1}{n}\right)\left(\frac{n-2}{n}\right) \cdots \left(\frac{n-r+1}{n}\right) . \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^r}$$

$$= \frac{\lambda^r}{r!} \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) \cdots \left(1-\frac{(r-1)}{n}\right) . \frac{\left[\left(1-\frac{\lambda}{n}\right)^{-n/\lambda}\right]^{-\lambda}}{\left(1-\frac{\lambda}{n}\right)^r}$$

as $n \to \infty$ each of the $(r-1)$ factors

$\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) \cdots \left(1-\frac{(r-1)}{n}\right)$ tends to 1

Also, $\left(1-\frac{\lambda}{n}\right)^r$ tends to 1

Since, $\lim_{x \to \pm\infty} \left(1+\frac{1}{x}\right)^x = e$

$\therefore \left[\left(1-\frac{\lambda}{n}\right)^{-n/\lambda}\right]^{-\lambda} = e^{-\lambda}$ as $n \to \infty$

hence, in limiting case when $n \to \infty$ we have,

$$\boxed{P(X=r) = \frac{e^{-\lambda} \cdot \lambda^r}{r!}} \qquad r = 0, 1, 2, 3, \ldots$$

where λ is a finite number $= np$.

<u>Note</u>: The sum of the probabilities $P(r)$ for $r = 0, 1, 2, 3 \ldots$ is 1.

$P(0) + P(1) + P(2) + P(3) + \cdots$

$= e^{-\lambda} + \frac{\lambda e^{-\lambda}}{1!} + \frac{\lambda^2 e^{-\lambda}}{2!} + \frac{\lambda^3 e^{-\lambda}}{3!} + \cdots$

$= e^{-\lambda}\left[1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots\right]$

$= e^{-\lambda} \cdot e^{\lambda} = 1$

Reccurrence Formula for the Poisson distribution

$$P(r) = \frac{e^{-\lambda} \cdot \lambda^{r}}{r!} \quad , \quad P(r+1) = \frac{e^{-\lambda} \cdot \lambda^{r+1}}{(r+1)!}$$

$$\therefore \frac{P(r+1)}{P(r)} = \frac{e^{-\lambda} \cdot \lambda^{r+1}}{(r+1)!} \cdot \frac{r!}{e^{-\lambda} \cdot \lambda^{r}} = \frac{\lambda}{(r+1)}$$

$$\boxed{P(r+1) = \frac{\lambda}{r+1} \cdot P(r)}$$

imp

Moment generating function of Poisson distribution

$$M_x(t) = \Sigma e^{tx} f(x)$$

$$= \Sigma e^{tx} \cdot \frac{e^{-\lambda} \cdot \lambda^{x}}{x!}$$

On Solving, $M_x(t) = e^{\lambda(e^t - 1)}$

$$\mu_1' = \left| \frac{d}{dt} M_x(t) \right|_{t=0} = \lambda$$

Mean $= \mu_1'$

$\Rightarrow$ $\boxed{\text{Mean} = \lambda}$

$$\mu_2' = \left| \frac{d^2}{dt^2} M_x(t) \right|_{t=0} = \lambda^2 + \lambda$$

Variance $= \mu_2 = \mu_2' - (\mu_1')^2 = \lambda$

$\Rightarrow$ $\boxed{\text{Variance} = \lambda}$

Ques If the variance of the Poisson distribution is 2, find the probabilities for $r=1,2,3,4$ from the recurrence relation of the Poisson distribution. Also, find $P(r \geq 4)$

Sol^n given: Variance $= \lambda = 2$

Recurrence relation for Poisson distribution is

$$P(r+1) = \frac{\lambda}{r+1} P(r) = \frac{2}{r+1} \cdot P(r) \quad \text{——— (1)}$$

From Poisson distribution

$$P(X=r) = \frac{e^{-\lambda} \cdot \lambda^r}{r!}$$

for $r=0$, $P(0) = \frac{e^{-\lambda} \cdot \lambda^0}{0!} = e^{-2} = 0.1353$

put $r=0,1,2,3$ in eq^n (1)

$r=0$, $P(1) = \frac{2}{1} \cdot P(0) = 2 \times 0.1353 = 0.2706$

$r=1$, $P(2) = \frac{2}{2} \cdot P(1) = 0.2706$

$r=2$, $P(3) = \frac{2}{3} P(2) = \frac{2}{3} \times 0.2706 = 0.1804$

$r=3$, $P(4) = \frac{2}{4} P(3) = \frac{1}{2} \times 0.1804 = 0.0902$

$$\begin{aligned} P(r \geq 4) &= P(4) + P(5) + P(6) + \cdots \\ &= 1 - [P(0) + P(1) + P(2) + P(3)] \\ &= 1 - [0.1353 + 0.2706 + 0.2706 + 0.1804] \\ &= 0.1431 \end{aligned}$$

2015
Ques Using Poisson distribution, find the probability that the ace of spades will be drawn from a pack of well-shuffled cards atleast once in 104 consecutive trials.

Sol.) probability of getting ace of spades, $p = \frac{1}{52}$

$$n = 104$$

$$\lambda = np = 104 \times \frac{1}{52} = 2$$

Probability of getting atleast once is

$$P(r \geq 1) = 1 - P(0) \qquad \left[P(r) = \frac{e^{-\lambda} \cdot \lambda^r}{r!}\right]$$

$$= 1 - \frac{e^{-\lambda} \cdot \lambda^0}{0!}$$

$$= 1 - e^{-2}$$

$$= 1 - 0.1353 = 0.8647$$

2017
Ques In a certain factory turning out razor blades, there is a small chance of 0.002 for any blade to be defective. The blades are supplied in packets of 10. Use suitable distribution to calculate the approximate number of packets containing no defective, one defective and two defective blades respectively in a consignment of 50,000 packets. (given: $e^{-0.02} = 0.9802$)

Sol.)

$$p = 0.002, \quad n = 10$$

$$\lambda = np = 10 \times 0.002 = \cancel{0.002} \; 0.02$$

No. of packets in consignment, $N = 50,000$

using Poisson distribution,

$$P(r) = \frac{e^{-\lambda} \cdot \lambda^r}{r!} = \frac{e^{-0.02} \cdot (0.02)^r}{r!}$$

(i) no defective : $r = 0$.

$$P(0) = \frac{e^{-0.02} \times (0.02)^0}{0!} = 0.9802$$

$\therefore$ approx. number of packets having zero defective in the consignment

$= 0.9802 \times 50,000$

$= 49010$

(ii) one defective : $r = 1$

$$P(1) = \frac{e^{-0.02} \times (0.02)^1}{1!} = 0.019604$$

$\therefore$ approx. number of packets having one defective blade $= 0.019604 \times 50,000$

$= 980.2 \approx 980$

(iii) two defective : $r = 2$

$$P(2) = \frac{e^{-0.02} \times (0.02)^2}{2!} = 0.000196$$

$\therefore$ approx. number of packets having two defective blade $= 0.000196 \times 50,000$

$= 9.8 \approx 10$

Ques The distribution of the number of road accidents per day in a city is Poisson with mean 4. Find the number of days out of 100 when there will be

(i) no accident (ii) atleast 2 accident

(iii) atmost 3 accidents

(iv) between 2 and 5 accidents.

Sol.: given: mean $= \lambda = 4$

No. of days, $N = 100$

$$P(r) = \frac{e^{-\lambda} \cdot \lambda^r}{r!} = \frac{e^{-4} \cdot 4^r}{r!}$$

(i) no accident, $r = 0$

$$P(0) = \frac{e^{-\lambda} \cdot \lambda^0}{0!} = e^{-4} = 0.01831$$

$\therefore$ no. of days with no accident $= 0.01831 \times 100$

$= 1.831 \approx 2$

(ii) atleast 2 accident: $r \geq 2$

$$P(r \geq 2) = P(2) + P(3) + P(4) + \cdots$$

$$= 1 - [P(1) + P(0)]$$

$$= 1 - \left[\frac{e^{-4} \cdot 4^1}{1!} + \frac{e^{-4} \cdot 4^0}{0!}\right]$$

$$= 1 - e^{-4}(4+1)$$

$$= 1 - 5e^{-4} = 0.90842$$

$\therefore$ no. of days having atleast two accidents

$= 0.90842 \times 100$

$= 90.842 \approx 91$

(iii) atmost 3 accidents: $r \leq 3$

$$P(r \leq 3) = P(0) + P(1) + P(2) + P(3)$$

$$= \frac{e^{-4} \cdot 4^0}{0!} + \frac{e^{-4} \cdot 4^1}{1!} + \frac{e^{-4} \cdot 4^2}{2!} + \frac{e^{-4} \cdot 4^3}{3!}$$

$$= e^{-4} + 4e^{-4} + 8e^{-4} + \frac{64}{6}e^{-4} = 0.43347$$

$\therefore$ no. of days having atmost three accidents

$= 0.43347 \times 100 = 43.347 \approx 43$

(iii) between 2 & 5 accidents : $2 < r < 5$

$$P(2<r<5) = P(3) + P(4)$$
$$= \frac{e^{-4} \cdot 4^3}{3!} + \frac{e^{-4} \cdot 4!}{4!}$$
$$= \left(\frac{64}{6} + \frac{256}{24}\right) e^{-4} = 0.3907$$

$\therefore$ no. of days having accidents between 2 and 5 $= 0.3907 \times 100 = 39.07 \approx 39$

Ques An insurance company insures 4000 people against loss of both eyes in a car accident. Based on previous data it was assumed 10 persons out of 1,00,000 will have such type of injury in car accident. What is the probability that more than 2 of the insured will collect on their policy in a given year?

Sol^n given, $n = 4000$

$$p = \frac{10}{1,00,000} = \frac{1}{10,000}$$

$$\lambda = np = 4000 \times \frac{1}{10,000} = \frac{4}{10} = 0.4$$

Probability for more than 2 i.e.

$$P(r>2) = P(3) + P(4) + P(5) + \cdots$$
$$= 1 - [P(0) + P(1) + P(2) + \cdots]$$
$$= 1 - \left[\frac{e^{-0.4}(0.4)^0}{0!} + \frac{e^{-0.4} \times (0.4)^1}{1!} + \frac{e^{-0.4} \times (0.4)^2}{2!}\right]$$
$$= 1 - e^{-0.4}[1 + 0.4 + 0.08]$$
$$= 0.008$$

2019

Ques Fit a Poisson distribution to the following data which give the number of yeast cells per square for 400 squares

X	0	1	2	3	4	5	6	7	8	9	10
F	103	143	98	42	8	4	2	0	0	0	0

It is given that $e^{-1.32} = 0.2674$

Sol.^{n} Mean of the given distribution $= \frac{\Sigma fx}{\Sigma f}$

$\therefore$ Mean $= \frac{143 + 196 + 126 + 32 + 20 + 12}{400} = \cancel{1.57}\ 1.32$

Mean $= \lambda = 1.57$

Required Poisson distribution $= N \cdot \frac{e^{-\lambda} \cdot \lambda^r}{r!} = N \cdot P(r)$

$= \frac{400 \cdot e^{-1.32} \cdot (1.32)^r}{r!}$

$= \frac{106.96 \times (1.32)^r}{r!}$

r	$N \cdot P(r)$		Theoretical Frequency
0	$\frac{106.96 \times (1.32)^0}{0!}$	$= 106.96$	107
1	$\frac{106.96 \times (1.32)^1}{1!}$	$= 141.18$	141
2	$\frac{106.96 \times (1.32)^2}{2!}$	$= 93.18$	93
3	$\frac{106.96 \times (1.32)^3}{3!}$	$= 41.0007$	41
4	$\frac{106.96 \times (1.32)^4}{4!}$	$= 13.53$	14
5	$\frac{106.96 \times (1.32)^5}{5!}$	$= 3.57$	4
6	$\frac{106.96 \times (1.32)^6}{6!}$	$= 1.176$	1
7		$= 0.1481$	0
8		$= 0.0244$	0
9		$= 0.0035$	0
10		$= 0.00047$	0
			391

2014
Ques Fit a Poisson distribution to the following data and calculate theoretical frequencies.

Deaths	0	1	2	3	4
Frequencies	122	60	15	2	1

2011
Ques It is given that 2% of the electric bulbs manufactured by a company are defective. Using Poisson distribution, find the probability that a sample of 200 bulbs will contain
(i) no defective bulb
(ii) two defective bulbs
(iii) at the most three defective bulbs.

[Ans: (i) 0.018315 (ii) 0146525 (iii) 0.43347]

Normal Distribution

The normal distribution is a continous distribution. It is a limiting case of binomial distribution when 'n' becomes very large and 'p' the probability of success is close to 1/2. The general equation of normal distribution is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

where x is the random variable s.t. $-\infty < x < \infty$

$f(x)$ is the probability density function of the normal distribution.

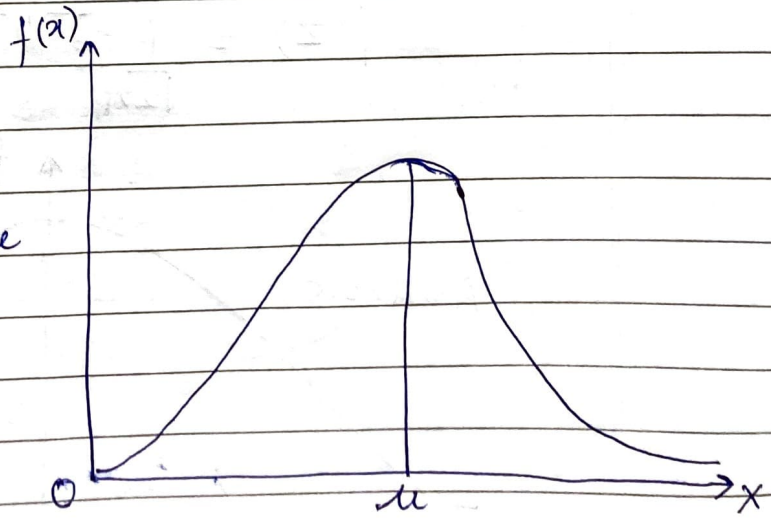
(i) $f(x) \geq 0$ (ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

σ is the standard deviation of the distribution such that $\sigma > 0$

μ is the mean of the distribution such that $-\infty < \mu < \infty$

The normal distribution is given by $N(\mu, \sigma^2)$

The graph of the normal distribution is called normal curve. It is a bell shaped and symmetrical about the mean μ. The two tails of the curve extend to $+\infty$ and $-\infty$ without ever meeting it. The line $x = \mu$ divides the



area under the normal curve above x-axis into two equal parts. So The curve is unimodal (i.e. unimodal distribution). The mean, mode and median of this distribution coincide.

Note:*

* The area under the normal curve between any two given ordinates $x = x_1$ and $x = x_2$ represents the probability of values falling into the given interval.
* The total area under the normal curve above the x-axis is 1.

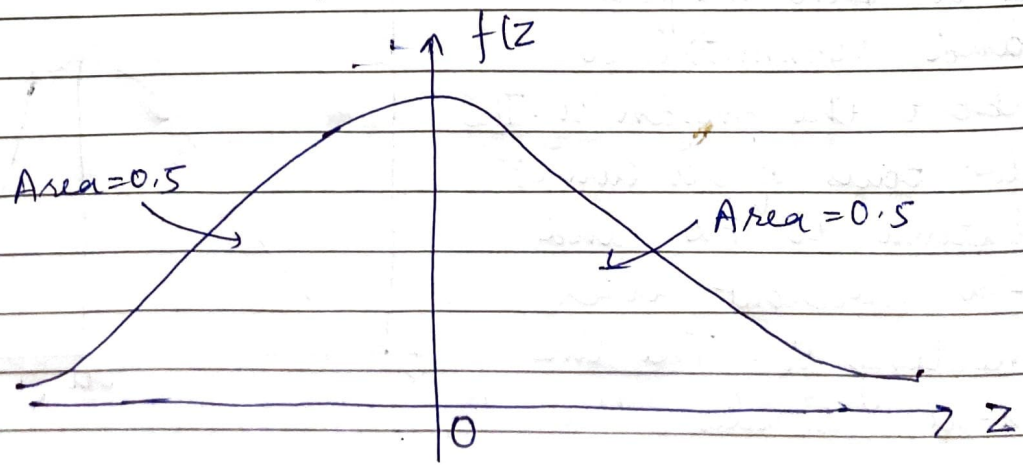
Standard form of the Normal Distribution

A random variable x with mean μ and standard deviation σ can be transformed to a standard normal variable 'z' s.t.

$$\boxed{z = \frac{x - \mu}{\sigma}}$$

then, the probability density function in the standard form is given by

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$



With the help of this formula, we can find the area under the normal probability curve by making use of standard tables.

Note

* If $f(z)$ is the probability density function for the normal distribution then,

$$P(z_1 \le z \le z_2) = \int_{z_1}^{z_2} f(z)\,dz = F(z_2) - F(z_1)$$

where $F(z) = \int_{-\infty}^{z} f(z)\,dz = P(z \le z)$

* The probabilities $P(z_1 \le z \le z_2)$, $P(z_1 < z \le z_2)$, $P(z_1 \le z < z_2)$ & $P(z_1 < z < z_2)$ are all same.

* $F(-z_1) = 1 - F(z_1)$

2017

Ques A sample of 100 dry battery cells tested to find the length of life produced the following results:

$\bar{x} = 12$ hours, $\sigma = 3$ hours

Assuming the data to be normally distributed, what percentage of battery cells are expected to have life:

(i) more than 15 hours (ii) less than 6 hours

(iii) between 10 and 14 hours?

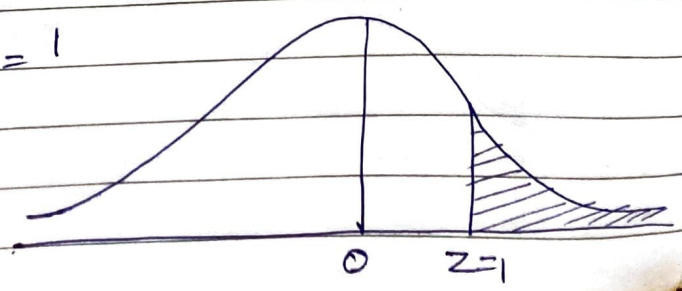
Sol^n Let x denote the length of life of dry battery cells

$$z = \frac{x - \bar{x}}{\sigma} = \frac{x - 12}{3}$$

(i) more than 15 hours. i.e. $x \ge 15$

$$z = \frac{15 - 12}{3} = 1$$

$P(x > 15) = P(z > 1)$



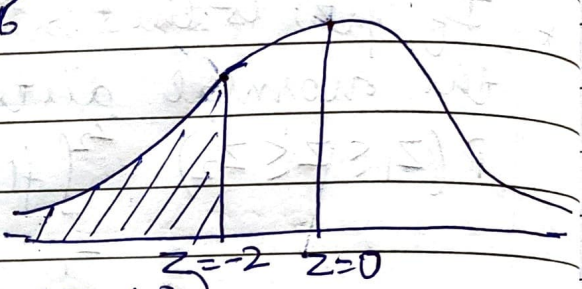
$= P(0 < Z < \infty) - P(0 < Z < 1)$

$= 0.5 - 0.3413 = 0.1587$

% of cells $= 0.1587 \times 100 = 15.87\%$

(ii) less-than 6 homes i.e. $x \leq 6$

$Z = \frac{6-12}{3} = -2$



$P(x<6) = P(Z<-2) = P(Z>2)$

$= P(0<Z<\infty) - P(0<Z<2)$

$= 0.5 - 0.4772$

$= \cancel{0.0228}\ 0.0228$

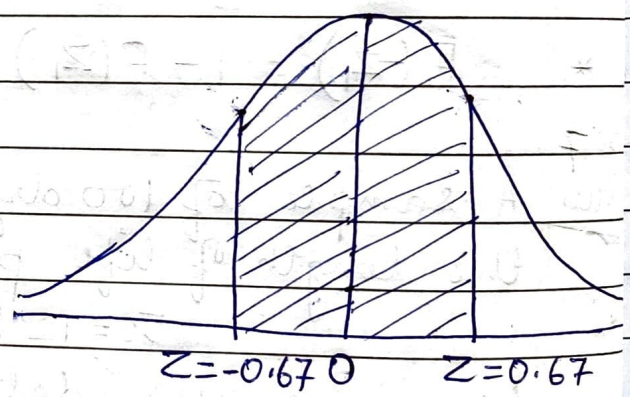
% of cells $= 0.0228 \times 100 = 2.28\%$

(iii) between 10 and 14 homes

i.e. $10 < x < 14$

When $x = 10$, $Z = \frac{10-12}{3} = -0.67$

When $x = 14$, $Z = \frac{14-12}{3} = 0.67$



$P(10 < x < 14) = P(-0.67 < Z < 0.67)$

$= 2P(0 < Z < 0.67)$

$= 2 \times 0.2485 = 0.4970$

% of cells $= 0.497 \times 100$

$= 49.70\%$

2019

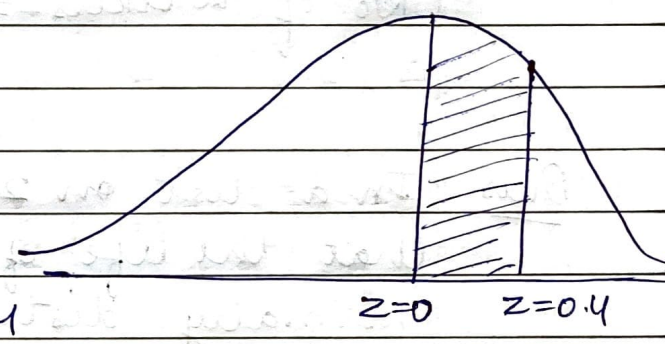
Ques The daily wages of 1000 workers are distributed around a mean of Rs. 140 and with a standard deviation of Rs. 10. Estimate the number of workers whose daily wages will be
(i) between Rs. 140 & Rs. 144 (ii) less than Rs. 126
(iii) more than Rs. 160.

Sol^n given $\bar{x} = 140$, $\sigma = 10$, $N = 1000$

$$z = \frac{x - \bar{x}}{\sigma} = \frac{x - 140}{10}$$

(i) between 140 & 144
i.e. $140 < x < 144$

When $x = 140$, $z = \frac{140-140}{10} = 0$



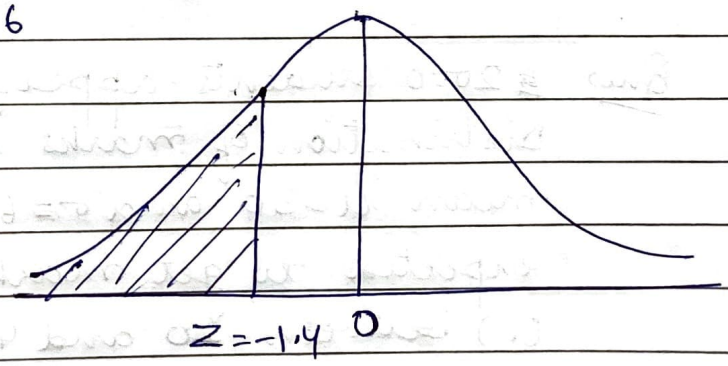
When $x = 144$, $z = \frac{144-140}{10} = 0.4$

$P(140 < x < 144) = P(0 < z < 0.4)$
$= 0.1554$

No. of workers $= 0.1554 \times 1000$
$= 155.4 = 155$

(ii) less than 126 i.e. $x < 126$

$z = \frac{126-140}{10} = -1.4$



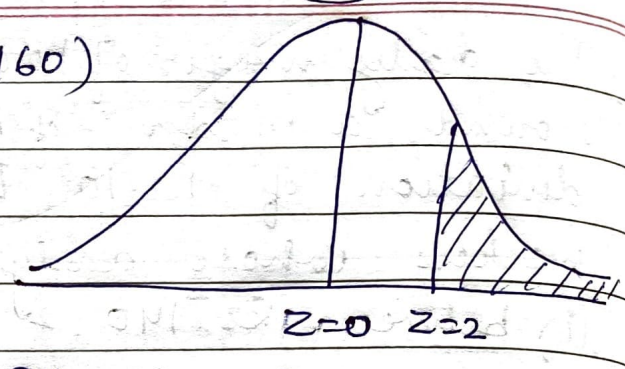
$P(x < 126) = P(z < -1.4)$
$= P(z > 1.4)$

$= P(0 < z < \infty) - P(0 < z < 1.4)$
$= 0.5 - 0.4192$
$= 0.0808$

No. of workers $= 0.0808 \times 1000 = 80.8$
$= 81$

(iii) more than 160 i.e. $(x > 160)$

$$z = \frac{160 - 140}{10} = 2$$



$$P(x > 160) = P(Z > 2)$$

$$= P(0 < Z < \infty) - P(0 < Z < 2)$$

$$= 0.5 - 0.4772$$

$$= 0.0228$$

No. of workers $= 0.0228 \times 1000 = 22.8$

$= 23$

Ques In a test on 2000 electric bulbs, it was found that the life of a particular make, was normally distributed with an average life of 2040 hours and S.D. of 60 hours, estimate the number of bulbs likely to burn for

(i) more than 2150 hours (ii) less than 1950 hours

(iii) more than 1920 hours but less than 2160 hours.

[Ans: (i) 67 (ii) 134 (iii) 1909]

Ques 2000 students appeared in an examination. Distribution of marks is assumed to be normal with mean $\mu = 30$ and $\sigma = 6.25$. How many students are expected to get marks

(i) between 20 and 40 (ii) less than 35

(iii) above 50

[Ans: (i) 1781 (ii) 1576 (iii) 1]

2016

Ques The life of army shoes is normally distributed with mean 8 months and standard deviation 2 months. If 5000 pairs are insured, how many pairs would be expected to need replacement after 12 months?

Sol^n given: $\bar{x} = 8$, $\sigma = 2$, $N = 5000$

$$Z = \frac{x - \bar{x}}{\sigma} = \frac{x-8}{2}$$

we have to find probability for $x > 12$

$$z = \frac{12-8}{2} = 2$$

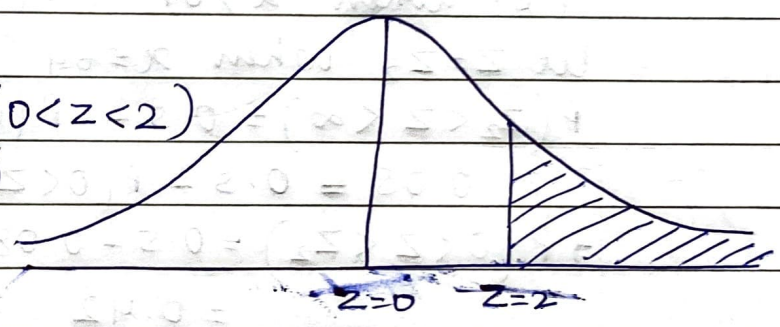
$\therefore P(x > 12) = P(z > 2)$

$= P(0 < z < \infty) - P(0 < z < 2)$

$= 0.5 - P(0 < z < 2)$

$= 0.5 - 0.4772$

$= 0.0228$



$\therefore$ No. of pairs whose life is more than 12 months

$= 5000 \times 0.0228 = 114$

$\therefore$ Pairs of shoes needing replacement after 12 months $= 5000 - 114 = 4886$

2018 Imp

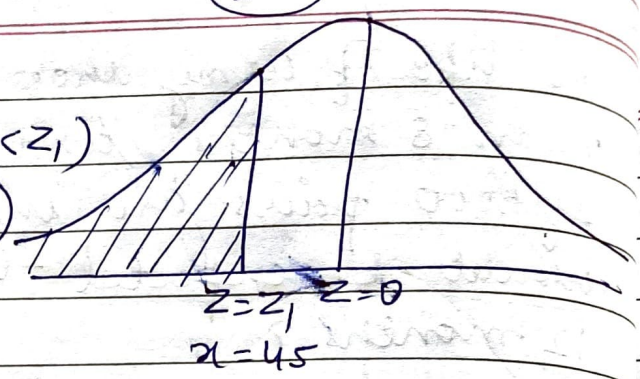
Ques In a normal distribution, 31% of the items are under 45 and 8% are over 64. Find the mean and standard deviation of the distribution. It is given that $f(0.5) = 0.19$ & $f(1.4) = 0.42$

Sol^n let $\bar{x}$ be the mean and σ be the S.D.

$$Z = \frac{x - \bar{x}}{\sigma}$$

(i) given 31% of items are under 45

i.e. When $x < 45$, $P(x < 45) = 0.31$



Let $z = z_1$ when $x = 45$

$P(-\infty < z < z_1) = 0.5 - P(0 < z < z_1)$

$0.31 = 0.5 - P(0 < z < z_1)$

$P(0 < z < z_1) = 0.5 - 0.31$

$= 0.19$

When we check from table

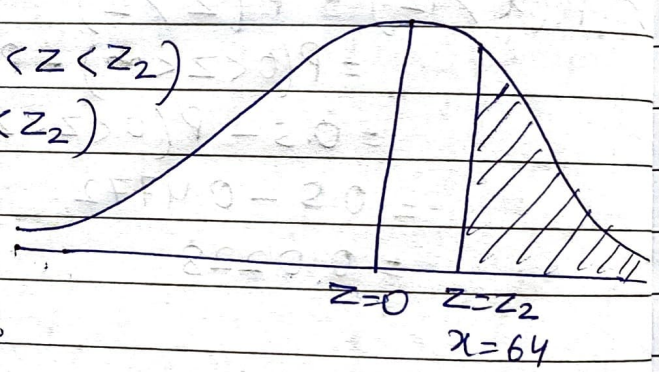
we get, $z_1 = 0.5$

as $z_1 < 0 \Rightarrow z_1 = -0.5$

(ii) again given 8% are over 64

i.e. when $x > 64$ $P(x > 64) = 0.08$

Let $z = z_2$ when $x = 64$



$P(z_2 < z < \infty) = 0.5 - P(0 < z < z_2)$

$0.08 = 0.5 - P(0 < z < z_2)$

$\Rightarrow P(0 < z < z_2) = 0.5 - 0.08$

$= 0.42$

When we check from table

we get $z_2 = 1.4$

as $z = \dfrac{x - \mu}{\sigma}$

$-0.5 = \dfrac{45 - \mu}{\sigma}$, $1.4 = \dfrac{64 - \mu}{\sigma}$

$45 - \mu = -0.5\sigma$, $64 - \mu = 1.4\sigma$

or, $-0.5\sigma + \mu = 45$, $1.4\sigma + \mu = 64$

Solving them we get

$\sigma = 10$, $\mu = 50$.

Ques The income of a group of 10,000 persons was found to be normally distributed with mean ₹ 750 p.m. and standard deviation of ₹ 50 ~~p.m~~. Show that of this group, about 95% had income exceeding ₹ 668 and only 5% had income exceeding ₹ 832.

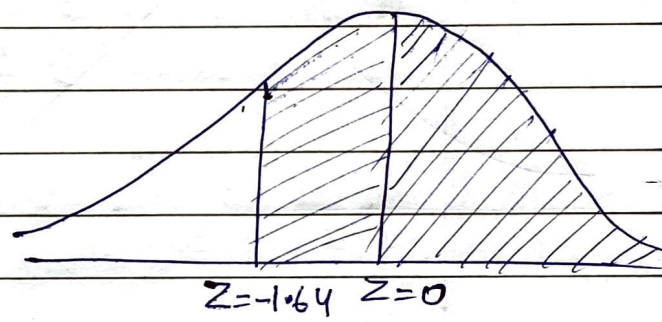
Sol: given: $\mu = 750$, $\sigma = 50$

$$z = \frac{x-\mu}{\sigma} = \frac{x-750}{50}$$

(i) % of people whose income exceed ₹ 668 i.e. $x > 668$

$$Z = \frac{668-750}{50} = -1.64$$

$$P(x > 668) = P(Z > -1.64)$$



$$= 0.5 + P(-1.64 < Z < 0)$$

$$= 0.5 + P(0 < Z < 1.64)$$

$$= 0.5 + 0.4495$$

$$= 0.9495$$

∴ Required % of people $= 0.9495 \times 100 = 94.95\% = 95\%$ app.

(ii) % of people whose income exceed ₹ 832 i.e. $x > 832$

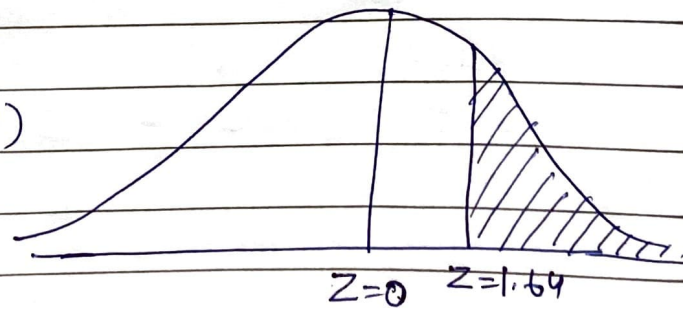
$$Z = \frac{832-750}{50} = 1.64$$

$$P(x > 832) = P(Z > 1.64)$$

$$= 0.5 - P(0 < Z < 1.64)$$

$$= 0.5 - 0.4495$$

$$= 0.0505$$



∴ Required % of people

$$= 0.0505 \times 100 = 5.05\%$$

$$= 5\% \text{ approx.}$$

Probability –

(1) Exhaustive events: The total number of all possible outcomes in any trial is known as exhaustive events or cases

eg: In tossing a coin, there are two exhaustive cases, head and tail.

In throwing a dice, there are six exhaustive cases, 1, 2, 3, 4, 5 or 6.

(2) Equally likely events: The given events are said to be equally likely, if none of them is expected to occur in preference to the other.

(3) Mutually Exclusive Events: Two events E_1 and E_2 are said to be mutually exclusive if $E_1 \cap E_2 = \phi$. The events which are not mutually exclusive are known as compatible events.

eg: In throwing of a dice, we have $S = \{1, 2, 3, 4, 5, 6\}$

Let E_1 = Event of getting any number less than 3.

E_2 = Event of getting any number more than 4

$\therefore E_1 = \{1, 2\}$ & $E_2 = \{5, 6\}$

so, $E_1 \cap E_2 = \phi$

i.e. E_1 & E_2 are mutually exclusive events.

(4) Independent events: Two events are said to be independent if the occurence of one does not depend upon the occurence of the other.

The events which are not independent are known as dependent events.

eg: let E_1 = event of occurrence of head on 1st coin.
E_2 = event of occurrence of head on 2nd coin.
Clearly, the occurrence of head on second coin does not depend upon the occurrence of head on first coin.
So, E_1 & E_2 are independent events.

Mathematical Definition of Probability

If a trial results in 'n' exhaustive, mutually exclusive and equally likely cases and 'm' of them are favourable to the happening of an event E, then the probability of happening of E is given by

$$P(E) = \frac{\text{Favourable number of cases}}{\text{Exhaustive number of cases}} = \frac{m}{n}$$

* Since the number of cases favourable to happening of E is m and the exhaustive number of cases is n, so, the number of cases unfavourable to happening of E are n−m.
i.e. $\bar{E} = n-m$

$$P(\bar{E}) = \frac{\text{unfavourable number of cases}}{\text{Exhaustive number of cases}} = \frac{n-m}{n}$$

* $P(E) + P(\bar{E}) = 1$

Additive law of Probability

If $E_1, E_2, E_3, \ldots E_n$ be n mutually exclusive events and $P(E_1), P(E_2), P(E_3) \ldots P(E_n)$ be their respective probabilities then,

$$P(E_1 + E_2 + E_3 + \cdots + E_n) = P(E_1) + P(E_2) + P(E_3) + \cdots + P(E_n)$$

Note: If two events E_1 and E_2 are not mutually exclusive then the probability of the event that either E_1 or E_2 or both will occur is given by

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) + P(E_1 \cap E_2)$$

Conditional Probability

The probability of the happening of an event E_1 when another event E_2 is known to have already happened is called Conditional Probability and is denoted by $P(E_1/E_2)$

Baye's Theorem

If $E_1, E_2, E_3 \ldots E_n$ are mutually exclusive and exhaustive events with $P(E_i) \neq 0$ $(i = 1, 2, \ldots n)$ of a random experiment then for any arbitrary event A of the sample space of the above experiment with $P(A) > 0$,

$$P(E_i/A) = \frac{P(E_i) \cdot P(A/E_i)}{\sum_{i=1}^{n} P(E_i) \cdot P(A/E_i)}$$

Ques A bag X contains 2 white and 3 red balls and a bag Y contains 4 white and 5 red balls. One ball is drawn at random from one of the bags and is found to be red. Find the probability that it was drawn from bag Y.

Sol^n Let E_1 – the ball is drawn from bag X

E_2 – the ball is drawn from bag Y.

A – the ball is red.

we have to find $P(E_2/A)$

$$P(E_2/A) = \frac{P(E_2) \cdot P(A/E_2)}{P(E_1)\,P(A/E_1) + P(E_2) \cdot P(A/E_2)} \quad \text{—— (1)}$$

Since the two bags are equally likely to be selected, $P(E_1) = P(E_2) = \frac{1}{2}$

$P(A/E_1) = P$(a red ball is drawn from bag X)

$= \frac{3}{5}$

$P(A/E_2) = P$(a red ball is drawn from bag Y)

$= \frac{5}{9}$

put the values in (1)

$$P(E_2/A) = \frac{\frac{1}{2} \times \frac{5}{9}}{\left(\frac{1}{2} \times \frac{3}{5}\right) + \left(\frac{1}{2} \times \frac{5}{9}\right)} = \frac{25}{52}$$

Ques The contents of urns I, II, and III are as follows:
1 white, 2 black and 3 red balls.
2 white, 1 black and 1 red balls and
4 white, 5 black and 3 red balls.
One urn is chosen at random and two balls drawn. They happen to be white and red. What is the probability that they come from urns I, II or III?

Sol: Let E_1 - urn I is chosen
E_2 - urn II is chosen
E_3 - urn III is chosen
A - The two balls are white & red.
We have to find $P(E_1/A)$, $P(E_2/A)$, $P(E_3/A)$

The three urns are equally likely to be selected
$P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$

$P(A/E_1) = P$(a white & red ball is chosen from urn I)
$= \frac{{}^1C_1 \times {}^3C_1}{{}^6C_2} = \frac{1}{5}$

$P(A/E_2) = P$(a white & red ball is chosen from urn II)
$= \frac{{}^2C_1 \times {}^1C_1}{{}^4C_2} = \frac{1}{3}$

$P(A/E_3) = P$(A white & red ball is chosen from urn III)
$= \frac{{}^4C_1 \times {}^3C_1}{{}^{12}C_2} = \frac{2}{11}$

by Baye's Theorem.

$$P(E_1/A) = \frac{P(E_1).P(A/E_1)}{P(E_1).P(A/E_1)+P(E_2).P(A/E_2)+P(E_3).P(A/E_3)}$$

$$= \frac{\frac{1}{3}\times\frac{1}{5}}{\left(\frac{1}{3}\times\frac{1}{5}\right)+\left(\frac{1}{3}\times\frac{1}{3}\right)+\left(\frac{1}{3}\times\frac{2}{11}\right)} = \frac{33}{118}$$

Similarly, $P(E_2/A) = \frac{55}{118}$

$P(E_3/A) = \frac{15}{59}$

Ques A factory has two machines A and B. Past record shows that machine A produced 60% of the items of output and machine B produced 40% of the items. Further 2% of the items produced by machine A were defective and 1% produced by machine B were defective. If a defective item is drawn at random, what is the probability that it was produced by machine A?

Sol.n Let E_1 – item is produced by machine A
E_2 – item is produced by machine B
A – The item is defective.

$P(E_1) = 0.6$, $P(E_2) = 0.4$

$P(A/E_1) = 0.02$, $P(A/E_2) = 0.01$

by Baye's theorem,

$$P(E_1/A) = \frac{P(E_1).P(A/E_1)}{P(E_1).P(A/E_1)+P(E_2).P(A/E_2)}$$

$$= \frac{0.6\times0.02}{0.6\times0.02+0.4\times0.01} = \frac{3}{4}$$

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Ques The probability that a civilian can hit a target is $\frac{2}{5}$ and the probability that an army officer can hit the same target is $\frac{3}{5}$. While the civilian can fire 8 shots in the time, the army officer fires 10 shots. If they fire together, then what is the probability that army officer shoots the target?

Sol^n Let E_1 – Civilian hit target

E_2 – army officer hit target

A – the target is shot.

$P(E_1) = \frac{2}{5}$, $P(E_2) = \frac{3}{5}$

$P(A/E_1) = 0.08$, $P(A/E_2) = 0.1$

by Baye's theorem,

$$P(E_2/A) = \frac{P(E_2) \cdot P(A/E_2)}{P(E_1) \cdot P(A/E_1) + P(E_2) \cdot P(A/E_2)}$$

$$= \frac{3/5 \times 0.1}{\frac{2}{5} \times 0.08 + \frac{3}{5} \times 0.1} = \frac{15}{23}$$

Ques An insurance company insured 2,000 scooter drivers, 4000 car drivers and 6,000 truck drivers. The probability of accident is 0.01, 0.03 & 0.15 respectively. One of the insured person meets an accident. What is the probability that he is a scooter driver?

$$\left[\text{Ans} = \frac{1}{52} \right]$$